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Mei Xiaochun

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The real Gama Function $\Gamma(a)$ is defined in the field of real number with $a > 0$. It is infinite and meaningless in the field with $a < 0$. However, according to the continuation theory of function at present, except at the points $a = 0, -1, -2, -3, \cdots$, $\Gamma(a)$ can be extended to the field of negative number with $a < 0$. It is pointed out that this continuation leads to infinite and is meaningless. The reason is that the form of Gama function has no any change in the extended field, so that contradiction is caused. It is proved that the single complex continuation of Gama function does not satisfy the Cauchy-Riemann equation, so it is not an analytic function. But the double complex continuation of the Gama function satisfies the Cauchy-Riemann equation and is an analytic function. Also, it is proved that the complex continuation of the complementary formula of the Gama function is incorrect. The correct result is calculated in the paper. The influences of these results on the Riemann Zeta function equation and the Riemann hypotheses are discussed.

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The real Gama Function $\Gamma(a)$ *is defined in the field of real number with* $a>0$. *It is infinite and meaningless in the field with* $a<0$. *However, according to the continuation theory of function at present, except at the points* $a=0,\ -1,\ -2,\ -3,\cdots$, $\Gamma(a)$ *can be extended to the field of negative number with* $a<0$. *It is pointed out that this continuation leads to infinite and is meaningless. The reason is that the form of Gama function has no any change in the extended field, so that contradiction is caused. It is proved that the single complex continuation of Gama function does not satisfy the Cauchy-Riemann equation, so it is not an analytic function. But the double complex continuation of the Gama function satisfies the Cauchy-Riemann equation and is an analytic function. Also, it is proved that the complex continuation of the complementary formula of the Gama function is incorrect. The correct result is calculated in the paper. The influences of these results on the Riemann Zeta function equation and the Riemann hypotheses are discussed. According to the correct formula, the zeros of Zeta function equation are located on the points* $a=\pm 1/2,\ \pm 3/2,\ \pm 5/2\cdots$, *so the Riemann hypothesis does not hold. The complex continuation of the Euler formula of prime numbers is also discussed. It is proved that the real part of the complex continuation formula is different from the original Euler formula of prime numbers. Because the trigonometric functions are contained in it, which my be irrational numbers, the extended formula does not describe the relation between natural numbers and prime numbers again. Therefore, the complex continuation of the Euler formula of prime numbers is meaningless in number theory. We can not discuss the distribution of prime numbers based on it.*

Keywords: gama function, negative continuation of function, complex continuation of gama function, analytic function, residue theorem, euler formula of prime number, riemann hypothesis, riemann zeta function equation, cauchy-riemann equation.

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I. INTRODUCTION

It was revealed in the author's first paper [1] that there were several serious mistakes in the Riemann's original paper to deduce the Zeta function equation in 1859.

1. An integral item around the original point of the coordinate system was ignored when Riemann deduced the integral form of the Riemann Zeta function. This item is convergent when $\mathrm{Re}(s)>1$ but divergent when $\mathrm{Re}(s)<1$. The integral form of the Zeta function has not changed the divergence of its series form.

2. A summation formula was used in the deduction of the integral form of the Zeta function. The applicable condition of this formula is $x > 0$. At point $x = 0$, the formula is meaningless. However, the lower limit of the Zeta function's integral is $x = 0$, so the formula can not be used.
3. The formula $\theta(x) = \sqrt{x}\theta(1/x)$ of Jacobi function was used to prove the symmetry of the Zeta function equation. The applicable condition of this formula is also $x > 0$. Because the lower limit of the integral is $x = 0$, this formula can not be used too.
4. Due to these mistakes, the inconsistency is caused on the two sides of the Zeta function equation. On the real axis of complex plane, the function equation holds only at point $\mathrm{Re}(s) = 1/2$ $(s = a + ib)$. However, at this point, the Zeta function is infinite, rather than zero. At other points of the real axis with $a \neq 0$ and $b = 0$, the two sides of the Zeta function equation are contradictory. When one side is finite, another side may be infinite.

Therefore, the Riemann Zeta function equation does not hold, and the Riemann hypothesis is meaningless. This is the fundamental reason why the Riemann hypothesis has not been proved for so many years. At present, the zero calculations of the Zeta function are approximate due to the series with different orders are used. The analytic property of the Zeta function is destroyed so that the so-called zeros are not real ones.

In the author's second paper [2], the Zeta function equation is assumed to be correct, a simple and strict method is proposed to prove that the Riemann Zeta function equation has no trivial zero. The method is to divide the real part and the imaginary part of the Zeta function equation completely. By comparing the real part and the imaginary part individually, an equation set about a and b is obtained.

It is proved that the only solution of this equation set is $a = \pm 2n$ and $b = 0$. So $a = 1/2$ is not the non-trivial zero of the Zeta function equation. If $a \neq \pm 2n$, four equations are obtained. It is proved that these four equations are not independent of each other. To make them independent, the only way is to let $a = 1/2$ and $b = 0$. However, in this case, we have $\zeta(1/2,\ 0) \to \infty$, rather than zero. Finally, the comparing method of infinite series is used to prove that the Zeta function itself has no zero. Therefore, the Riemann hypothesis does not hold.

Because the Riemann Zeta function equation is based on the Gama function, the negative and complex continuations of the Gama function are discussed in this paper. To do it, we need to define the continuation of a function clearly at first. Assume that the function $f_1(x)$ has a clear definition in the field L_1 but it is meaningless in the field L_2. To make it meaningful in the field L_2, the function should be extended into $f_2(x)$. The analytic continuation of a function should satisfy the following conditions.

1. After the continuation, the form of the function $f_2(x)$ in the field L_2 should have something different from the function $f_1(x)$. Otherwise, the contradiction would be caused, and the continuation became meaningless.
2. In the original field L_1, the form of function $f_2(x)$ must be completely the same as the form of function $f_1(x)$, otherwise $f_2(x)$ can not be regarded as the continuation of $f_1(x)$.
3. Just for the sake of uniqueness, the continuation of function $f_1(x)$ must be analytic, or the derivative of function $f_2(x)$ exists everywhere. If it is a complex continuation, the Cauchy-Riemann equation must be satisfied.

The Gama function is a very important function with wide applications in pure mathematics and practical problems. Some people think that its importance is only next to the trigonometric function and the exponential function.

For example, based on the Gama function, the integral form of the Riemann Zeta function is introduced. It is the basis of Riemann hypothesis problem. It is proved in this paper that the negative continuation of the Gama function does not satisfy the condition 1 and leads to infinite. The single complex continuation of the Gama function does not satisfy the condition 3, so it is not an analytic function. But the double complex continuation of the Gama function satisfies all three conditions, so it is an analytic continuation.

It is proved that the present complex continuation $\Gamma(s)\Gamma(1-s)=\pi/\sin s\pi$ of the Gama function complementary formula is wrong. It is the result directly let $a\to s$ in $\Gamma(a)\Gamma(1-a)=\pi/\sin a\pi$. The correct calculation result is

$$\Gamma(s)\Gamma(1-s)=\frac{\pi e^{b\pi}}{\sin a\pi} \tag{1}$$

The result is a real number, rather than a complex number. The influences of these results on the Riemann Zeta function equation and Riemann hypotheses are discussed in this paper. It is proved that after Eq.(1) is used, the zeros of the Zeta function equation are located on the points $a=\pm 1/2,\ \pm 3/2,\ \pm 5/2\cdots$, the Riemann hypothesis is proved invalid from another angle.

The complex continuation of the Euler formula of prime numbers is discussed in this paper. It is proved that after the Euler formula of prime numbers is extended into the complex number field, the real part of the formula is different from the original Euler formula of prime numbers. Because it contains the trigonometric functions $\sin(b\ln p_j)$ and $\cos(b\ln p_j)$ which may be irrational numbers, the extended formula does not describe the relation between natural numbers and prime numbers again. Therefore, the complex continuation of the Euler prime number formula is meaningless in number theory. We can not discuss the distribution of prime numbers based on it.

II. THE CONTINUATIONS OF GENERAL FUNCTIONS

2.1 The analytic continuations of real functions

As we knew that a function has its definition field in general. Out of the field, the function may be meaningless. In order to make the function meaningful in the greater field, the continuation is needed. The common example of a real function's continuation is

$$f_1(x)=1+x+x^2+x^3+\cdots \qquad |x|<1 \tag{2}$$

Where $x\in R$ is a real number. Eq. (1) is meaningful and limited in the field $|x|<1$. when $|x|>1$, the function $f_1(x)\to\infty$ and becomes meaningless. On the other hand, we define

$$*\ f_2(x)=\frac{1}{1-x} \qquad x\neq 1 \tag{3}$$

$f_2(x)$ is meaningful on the whole number axis except at the point $x=1$. At all points of the field with $|x|<1$, we always have $f_1(x)=f_2(x)$. By developing $f_2(x)$ into the Taiwan's series when $|x|<1$, we can prove $f_1(x)=f_2(x)$ completely.

However, when $|x|>1$, Eqs. (2) and (3) can not be equal each other. For example, let $x=2$, we have $f_1(2)\to\infty$ and $f_2(2)=-1$. Because the definition field of $f_2(x)$ is greater than that of $f_1(x)$, we consider $f_2(x)$ as the continuation of $f_1(x)$ in the field with $|x|>1$ and write two functions in the unified form as below

$$f(x)=\begin{cases}1+x+x^2+x^3+\cdots & |x|<1 \\ \dfrac{1}{1-x} & |x|\neq 1\end{cases} \tag{4}$$

In general, a function can be extended in many different ways, resulting in different results. In order to ensure the uniqueness, the continuation of function needs to meet the continuity condition, so that the function can be differentiated everywhere. The continuation that meets this condition is called as the analytic continuation [3].

It is important to emphasize that at every point in the small field where the original function is meaningful, the value of the extended function should be exactly the same as the value of original function, otherwise it is not the continuation of the original function. Meanwhile, in the extended field, the form of the extended function must be different from the original function, otherwise the extension of the function is meaningless [3].

For example, for Eqs. (2) and (3), in the field $|x|<1$ where the original function makes sense, the values of function $f_2(x)$ and $f_1(x)$ must be the same at every point. Although their forms look different on the surface, they are the same actually. In the field $|x|>1$ after the continuation, the forms of $f_2(x)$ and $f_1(x)$ must be different. Otherwise $f_2(x)$ is still equal to $f_1(x)$, which is no meaning.

It is difficult to find the form of analytic continuation of a function in practical problems. The analytic continuations of some functions in existing mathematics are actually unsuccessful. As we see below, the negative continuation of the Gama function violates the above principles. In the extended field, the form of the function is exactly the same as the original function’s form, so it is still infinite and meaningless.

2.2 The analytic continuation of complex function

Let $z=x+iy\in C$ be a complex number, a similar example of analytic continuation of complex function is shown below.

$$F_1(z)=1-z^2+z^4-z^6+\cdots \qquad |z|<1$$

$$F_2(z)=\frac{1}{1+z^2} \qquad z\neq\pm i \tag{5}$$

$F_1(z)$ is an analytic function and convergent inside the unit circle but is divergent outside the unit circle without meaning. $F_2(z)$ is an analytic function meaningful on the whole complex plane except at points $z \neq \pm i$,

To developing $F_2(z)$ into the Taylor's series of complex functions in the field $|z| < 1$, we can obtain $F_1(z)$. That is to say, two functions are completely the same. Since the definition field of $F_2(z)$ is larger than $F_1(z)$, $F_2(z)$can be regarded as an analytic continuation of $F_1(z)$ over the entire complex plane (except at points $z \neq \pm i$). Similar to Eq.(4), we write they in the unified form

$$F(z) = \begin{cases} 1 - z^2 + z^4 - z^6 + \cdots & (|z| < 1) \\ \dfrac{1}{1+z^2} & (z \neq \pm i) \end{cases} \tag{6}$$

Similarly, within the extended field $|z| > 1$ ($z \neq \pm i$), $F_1(z)$and $F_2(z)$ are not the same functions, because $F_1(z)$ is meaningless in this case.

III. THE NEGATIVE CONTINUATION OF THE REAL GAMA FUNCTION

3.1 The definition and character of the real Gama function

The original definition of the real Gama function is

$$\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} dt \qquad a > 0 \tag{7}$$

Here parameter $a \in R$ is a real number with $a > 0$. If $a \leq 0$, the integral of Eq.(7) is infinite and meaningless. The Gama function has the following natures [3]

$$\Gamma(1) = 1 \qquad \Gamma(1/2) = \sqrt{\pi} \tag{8}$$

$$\Gamma(a+1) = a\Gamma(a) \quad \text{or} \quad \Gamma(a) = \frac{\Gamma(a+1)}{a} \tag{9}$$

$$\Gamma(n+1) = n! \qquad (n = 0,\ 1,\ 2,\ 3, \cdots) \tag{10}$$

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin a\pi} \tag{11}$$

$$\Gamma(2a) = 2^{2a-1}\pi^{-1/2}\Gamma(a)\Gamma(a+1/2) \tag{12}$$

According to the formulas above, the Gama function has no zero with $\Gamma(a) > 0$ in the field $a > 0$.

3. 2 The negative continuation of the Gama function does not hold.

According to the original definition, the Gama function is meaningless when $a \leq 0$. For example, when $a = 0$, Eq. (7) becomes

$$\Gamma(0)=\int_0^\infty e^{-t}t^{-1}dt=-\int_0^\infty t^{-1}de^{-t}=-t^{-1}e^{-t}\Big|_0^\infty+\int_0^\infty e^{-t}dt^{-1}$$

$$=-\frac{1}{e^t t}\Big|_0^\infty-\int_0^\infty e^{-t}t^{-2}dt=\infty-\Gamma(-1)\to\infty \tag{13}$$

By the same method, it can be proved that

$$\Gamma(-1)=\frac{\Gamma(1-1)}{-1}=\frac{\Gamma(0)}{-1}\to-\infty \tag{14}$$

$$\Gamma(-2)=\frac{\Gamma(1-2)}{-2}=\frac{\Gamma(-1)}{-2}\to\infty \tag{15}$$

Therefore, when $a=0,\ -1,\ -2,\ -3,\cdots$, we have $\Gamma(-a)\to\pm\infty$.

However, according to the present theory of the Gama function, as long as $a\neq 0,\ -1,\ -2,\ -3,\cdots$, by using formula (9), Eq.(7) can still be extended from the field $a>0$ to the field $a'=-a<0$, called the negative continuation of the Gama function.We write it as

$$\Gamma(a')=\Gamma(-a)=\frac{\Gamma(1-a)}{-a}\qquad a'\neq 0,\ -1,\ -2,\ -3,\cdots \tag{16}$$

For example, let $a'=-1/2$, according to Eqs. (7), (8) and (16), we have [3]

$$\Gamma(-1/2)=\frac{\Gamma(1-1/2)}{-1/2}=\frac{\Gamma(1/2)}{-1/2}=-2\sqrt{\pi} \tag{17}$$

Let $a'=-3/2$, according to Eqs.(16) and (17), we have

$$\Gamma(-3/2)=\frac{\Gamma(1-3/2)}{-3/2}=\frac{\Gamma(-1/2)}{(-3/2)}=\frac{4\sqrt{\pi}}{3} \tag{18}$$

So, according to the present theory, after negative continuation, as long as $a'=-a\neq 0,\ -1,\ -2,\ -3,\cdots$, the Gama function is still meaningful. The formula (16) appears in various textbooks and literature, and be used widely in mathematics.

However, according to this direct negative continuation, the basic form of the Gama function has no any change. It violates the first condition and is wrong. For example, to take $a'=-0.3$, according to the original definition Eq.(7) and Eq.(16), we have

$$\Gamma(-0.3)=-\frac{1}{0.3}\Gamma(1-0.3)=-\frac{1}{0.3}\Gamma(0.7)$$

$$=-\frac{1}{0.3}\int_0^\infty e^{-t}t^{0.7-1}dt=\frac{1}{0.3}\int_0^\infty t^{-0.3}de^{-t}$$

$$= \frac{1}{0.3}\frac{1}{t^{0.3}e^{t}}\bigg|_{0}^{\infty} - \frac{1}{0.3}\int_{0}^{\infty} e^{-t}dt^{-0.3}$$

$$= -\infty + \int_{0}^{\infty} e^{-t}t^{-0.3-1}dt = -\infty + \Gamma(-0.3) \qquad (19)$$

So we get the absurd result $\Gamma(-0.3) = -\infty + \Gamma(-0.3)$,the result is meaningless. So Eq.(16) can not hold. Again, to take $a' = -1.5$, according to Eq.(16), we have

$$\Gamma(-3/2) = \Gamma(-1.5) = -\frac{1}{1.5}\Gamma(1-1.5) = -\frac{1}{1.5}\Gamma(-0.5)$$

$$= -\frac{1}{1.5}\int_{0}^{\infty} e^{-t}t^{-0.5-1}dt = \frac{1}{1.5}\int_{0}^{\infty} t^{-1.5}de^{-t}$$

$$= \frac{1}{1.5}\frac{1}{t^{1.5}e^{t}}\bigg|_{0}^{\infty} - \frac{1}{1.5}\int_{0}^{\infty} e^{-t}dt^{-1.5}$$

$$= -\infty + \int_{0}^{\infty} e^{-t}t^{-1.5-1}dt = -\infty + \Gamma(-1.5) \qquad (20)$$

We still get $\Gamma(-1.5) = -\infty + \Gamma(-1.5)$, which is completely different from Eq.(18). In fact in the general situation, let $a > 0$, we have $-a < 0$. According to Eqs.(7) and (16), the result is

$$\Gamma(a') = \Gamma(-a) = -\frac{1}{a}\Gamma(1-a) = -\frac{1}{a}\int_{0}^{\infty} e^{-t}t^{1-a-1}dt$$

$$= \frac{1}{a}\int_{0}^{\infty} t^{-a}de^{-t} = \frac{1}{ae^{t}t^{a}}\bigg|_{0}^{\infty} - \frac{1}{a}\int_{0}^{\infty} e^{-t}dt^{-a}$$

$$= -\frac{1}{ae^{t}t^{a}}\bigg|_{t\to 0} + \int_{0}^{\infty} e^{-t}t^{-a-1}dt = -\infty + \Gamma(-a) \qquad (21)$$

That is $\Gamma(-a) = -\infty + \Gamma(-a)$. Therefore, the negative continuation formula (16) of the Gama function does not hold. The reason is that in the extend field, the form of the Gama function has not been changed.

IV. THE COMPLEX CONTINUATION OF THE GAMA FUNCTION

4.1 The single complex continuation and the double complex continuation of theGama function.

Let $a \to s = a + ib$ in Eq. (7), the Gama function is extended from real field to complex field with

$$\Gamma(s) = \int_{0}^{\infty} e^{-x}x^{s-1}dx \qquad \mathrm{Re}(s) > 0 \qquad (22)$$

We call Eq. (22) the single complex continuation of the Gama function. In the deduction of the integral form of the Riemann Zeta function, the single complex continuation of the Gama function is involved.

According to the definition, the condition that Eq.(22) holds is $\mathrm{Re}(s)=a>0$. That is to say, Eq.(22) holds only on the right half side of complex plane. On the left half side of complex plane, it does not hold. However, according to the present theory, Eq. (22) is actually considered to be valid on the whole complex plane except at the points of negative integers with $a=0,\ -1,\ -2,\ -3,\cdots$. The reason is due to the following relation

$$\Gamma(s+1)=s\Gamma(s) \tag{23}$$

Let $s'=-s$, we re-write Eq.(23) as

$$\Gamma(s')=\frac{\Gamma(s'+1)}{s'} \qquad \mathrm{Re}(s')\neq 0,\ -1,\ -2,\ \cdots \tag{24}$$

In this way, $\Gamma(s)$ is extended into the negative half plane with $\mathrm{Re}(s')<0$. Similar to the negative continuation of the real Game function as shown in Eq.(16), we call Eq. (24) the negative continuation of the single complex Game function.

By substituting $x^{ib}=e^{ib\ln x}$ in Eq.(22) and separating the real part and the imaginary part, we have

$$\begin{aligned}\Gamma(s)&=\int_0^\infty e^{-x}x^{a-1+ib}dx=\int_0^\infty e^{-x}x^{a-1}e^{ib\ln x}dx\\&=\int_0^\infty e^{-x}x^{a-1}\cos(b\ln x)dx+i\int_0^\infty e^{-x}x^{a-1}\sin(b\ln x)dx\\&=\Gamma_1(a,b)+i\,\Gamma_2(a,b)\end{aligned} \tag{25}$$

Here $\Gamma_1(a,b)$ and $\Gamma_2(a,b)$ are the real functions with

$$\Gamma_1(a,b)=\int_0^\infty e^{-x}x^{a-1}\cos(b\ln x)dx \qquad \Gamma_2(a,b)=\int_0^\infty e^{-x}x^{a-1}\sin(b\ln x)dx \tag{26}$$

It can be seen that $\Gamma_1(a,b)$ and $\Gamma_2(a,b)$ are not the originally defined Gama functions due to the existence of the items $\cos(b\ln x)$ and $\sin(b\ln x)$. A new parameter b is increased in Eq. (26), so it becomes the function of double parameters. We can not call them the Gama function again. The formulas (8) ~ (12) can not hold for them. They need to be proved again. Only when $b=0$, Eq.(25) degenerates to the real Gama function.

If let $x\to z=x+iy\in C$ again in Eq.(22), we call it the double complex continuation of the Gama function with

$$\Gamma_c(s)=\int_C e^{-z}z^{s-1}dz \tag{27}$$

In the deduction of the Riemann Zeta function equation, the double complex continuation of the Gama function is involved. We will discuss them in Section 4.4.

4. 2 The negative continuation of the complex Gama function does not hold.

We now prove that the formula (24) of the negative continuation of the complex Gama function does not hold on the complex half-plane $\mathrm{Re}(s') < 0$. Similarly, Eq.(24) can be written as

$$\begin{aligned}\Gamma(s') = \Gamma(-s) &= \frac{\Gamma(1-s)}{-s} = \frac{1}{-s}\int_0^\infty e^{-x}x^{1-s-1}dx \\ &= \frac{1}{s}\int_0^\infty x^{-s}de^{-x} = \frac{1}{s}e^{-x}x^{-s}\Big|_0^\infty - \frac{1}{s}\int_0^\infty e^{-x}dx^{-s} \\ &= \frac{1}{se^x x^s}\Big|_0^\infty + \int_0^\infty e^{-x}x^{-s-1}dx \end{aligned} \tag{28}$$

We have

$$x^s = x^{a+ib} = x^a e^{ib\ln x} = x^a\cos(b\ln x) + ix^a\sin(b\ln x) \tag{29}$$

When $x \to 0$, we have $\ln x \to -\infty$, but $\cos(b\ln x)$ and $\sin(b\ln x)$ are limited, so we have

$$\frac{1}{se^x x^s}\Big|_0^\infty = \frac{1}{(a+ib)e^x x^a[\cos(b\ln x) + i\sin(\ln x)]}\Big|_0^\infty \to \infty \tag{30}$$

From Eq.(28), we still obtain $\Gamma(-s) = \infty + \Gamma(-s)$, the negative continuation of the complex Gama function is also meaningless.

4. 3 The single complex continuation of the Gama function is not an analytic function

In the theory of complex function, the analytic property of function is very important. There are many theorems which are effective only for analytic functions, such as the Cauchy theorem, the residue theorem, and so on. These theorems do not work if the functions are not analytic ones.

To express the analytic nature, the real part and the imaginary part of complex function $f(s) = f(a+ib)$ should be separated. We write it as

$$f(s) = u(a,b) + iv(a,b) \tag{32}$$

The real part $u(a,b)$ and the imaginary part $v(a,b)$ are not independent, both should satisfy the following Cauchy-Riemann equation [4]

$$\frac{\partial u}{\partial a} = \frac{\partial v}{\partial b} \qquad \frac{\partial u}{\partial b} = -\frac{\partial v}{\partial a} \tag{33}$$

At present, the single complex continuation $\Gamma(s)$ of the real Gama function is considered an analytic function. But it is not true. We prove it below.

Let $u(a,b) = \Gamma_1(a,b)$ and $v(a,b) = \Gamma_2(a,b)$ in Eq.(25). If $\Gamma(\mathrm{s})$ is an analytic function, following relation should be satisfied.

$$\frac{\partial\Gamma_1}{\partial a}=\frac{\partial\Gamma_2}{\partial b} \qquad \frac{\partial\Gamma_2}{\partial a}=-\frac{\partial\Gamma_1}{\partial b} \tag{34}$$

However, according to Eq.(26), we have

$$\frac{\partial\Gamma_1}{\partial a}=(a-1)\int_0^\infty e^{-x}x^{a-2}\cos(b\ln x)dx \qquad \frac{\partial\Gamma_1}{\partial b}=-\int_0^\infty e^{-x}x^{a-1}\ln x\sin(b\ln x)dx \tag{35}$$

$$\frac{\partial\Gamma_2}{\partial a}=(a-1)\int_0^\infty e^{-x}x^{a-2}\sin(b\ln x)dx \qquad \frac{\partial\Gamma_2}{\partial b}=\int_0^\infty e^{-x}x^{a-1}\ln x\cos(b\ln x)dx \tag{36}$$

It is obvious that Eq.(34) can not hold, so the Gama function $\Gamma(s)$ described by Eq.(25) is not an analytic function on the complex plane $s=a+ib$.

4. 4 The double complex continuation of the Gama function is an analytic function.

We prove below that the double complex continuation of the Gama function is an analytic function. We write Eq.(27) as

$$\Gamma(s)=\int_C e^{-z}z^{s-1}dz=u(a,b)+iv(a,b) \tag{37}$$

We have

$$z^{s-1}=(re^{i\theta})^{a+ib-1}=(e^{\ln r}e^{i\theta})^{a-1+ib}=e^{(a-1)\ln r-b\theta}e^{i[b\ln r+(a-1)\theta]} \tag{38}$$

$$e^{-z}z^{s-1}=e^{-(x+iy)}z^{(a+ib)-1}=e^{(a-1)\ln r-b\theta-x}e^{i[b\ln r+(a-1)\theta-y]} \tag{39}$$

$$=e^{(a-1)\ln r-b\theta-x}\left\{\cos\left(b\ln r+(a-1)\theta-y\right)+i\sin\left(b\ln r+(a-1)\theta-y\right)\right\}=A+iB \tag{40}$$

$$A=e^{(a-1)\ln r-b\theta-x}\cos\left(b\ln r+(a-1)\theta-y\right) \tag{41}$$

$$B=e^{(a-1)\ln r-b\theta-x}\sin\left(b\ln r+(a-1)\theta-y\right) \tag{42}$$

Here $x=r\cos\theta$ and $y=r\sin\theta$. Then, we write Eq.(37) as

$$\Gamma(s)=\int_C e^{-z}z^{s-1}dz=\int_C(A+iB)(dx+idy)=\int_C(Adx-Bdy)+i\int_C(Ady+Bdx) \tag{43}$$

and get

$$u(a,b)=\int_0^\infty(Adx-Bdy) \qquad v(a,b)=\int_0^\infty(Ady+Bdx) \tag{44}$$

$$\frac{\partial u}{\partial a}=\int_C\left[\left(\frac{\partial A}{\partial a}\right)dx-\left(\frac{\partial B}{\partial a}\right)dy\right] \qquad \frac{\partial v}{\partial b}=\int_C\left[\left(\frac{\partial A}{\partial b}\right)dy+\left(\frac{\partial B}{\partial b}\right)dx\right] \tag{45}$$

$$\frac{\partial u}{\partial b}=\int_C\left[\left(\frac{\partial A}{\partial b}\right)dx-\left(\frac{\partial B}{\partial b}\right)dy\right] \qquad \frac{\partial v}{\partial a}=\int_C\left[\left(\frac{\partial A}{\partial a}\right)dy+\left(\frac{\partial B}{\partial a}\right)dx\right] \tag{46}$$

From Eq.(41) and (42), we have

$$\begin{aligned}\frac{\partial A}{\partial a}&=\ln r e^{(a-1)\ln r-b\theta-x}\cos\big(b\ln r+(a-1)\theta-y\big)\\ &-\theta\, e^{(a-1)\ln r-b\theta-x}\sin\big(b\ln r+(a-1)\theta-y\big)\end{aligned} \tag{47}$$

$$\begin{aligned}\frac{\partial B}{\partial a}&=\ln r\, e^{(a-1)\ln r-b\theta-x}\sin\big(b\ln r+(a-1)\theta-y\big)\\ &+\theta\, e^{(a-1)\ln r-b\theta-x}\cos\big(b\ln r+(a-1)\theta-y\big)\end{aligned} \tag{48}$$

$$\begin{aligned}\frac{\partial A}{\partial b}&=-\theta\, e^{(a-1)\ln r-b\theta-x}\cos\big(b\ln r+(a-1)\theta-y\big)\\ &-\ln r\, e^{(a-1)\ln r-b\theta-x}\sin\big(b\ln r+(a-1)\theta-y\big)\end{aligned} \tag{49}$$

$$\begin{aligned}\frac{\partial B}{\partial b}&=-\theta\, e^{(a-1)\ln r-b\theta-x}\sin\big(b\ln r+(a-1)\theta-y\big)\\ &+\ln r\, e^{(a-1)\ln r-b\theta-x}\cos\big(b\ln r+(a-1)\theta-y\big)\end{aligned} \tag{50}$$

By considering Eqs.(47) ~ (50), we see that Eq.(33) can be satisfied. So Eq.(44) satisfies the Cauchy-Riemann equation, the double complex continuation of the Gama function is an analytic function.

4.5 The product of the Riemann Zeta function and the Gama function is not an analytic function.

The original definition of the series form of the Riemann Zeta function is

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s} \qquad \operatorname{Re}(s)>1 \tag{51}$$

It is proved in [2] that Eq.(51) is an analytic function. Riemann obtained the relation between the Riemann Zeta function and the complex continuation of the Gama function in his original paper in 1859.

$$\zeta(s)\Gamma(s)=\int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx=I(s) \tag{52}$$

We prove below that $I(s)$ is not an analytic function about $s=a+ib$. By using $x^{ib}=e^{ib\ln x}$, Eq.(52) can be written as

$$I(s)=\int_0^\infty \frac{x^{a-1}e^{ib\ln x}}{e^x-1}dx=\int_0^\infty \frac{x^{a-1}}{e^x-1}\left[\cos(b\ln x)+i\sin(b\ln x)\right]dx \tag{53}$$

By separating the real part and the imaginary part, we get

$$u(a,b)=\int_0^\infty \frac{x^{a-1}}{e^x-1}\cos(b\ln x)dx \qquad v(a,b)=\int_0^\infty \frac{x^{a-1}}{e^x-1}\sin(b\ln x)dx \tag{54}$$

So we have

$$\frac{\partial u}{\partial a}=(a-1)\int_0^\infty \frac{x^{a-2}}{e^x-1}\cos(b\ln x)dx \qquad \frac{\partial u}{\partial b}=-\int_0^\infty \frac{x^{a-1}}{e^x-1}\ln x\sin(b\ln x)dx \tag{55}$$

$$\frac{\partial v}{\partial b}=\int_0^\infty \frac{x^{a-1}}{e^x-1}\ln x\cos(b\ln x)dx \qquad \frac{\partial v}{\partial a}=(a-1)\int_0^\infty \frac{x^{a-2}}{e^x-1}\sin(b\ln x)dx \tag{56}$$

It is obvious that Eqs.(55) and (56) are similar to Eqs.(35) and (36). The Cauchy-Riemann equation （33） can not be satisfied, so Eq. (52) is not an analytic function.

4. 6 The double complex continuation of $I(\mathrm{s})$ function is an analytic function.

Similar to Eq.(27), let $x\to z=x+iy$ in Eq.(52) for the second complex continuation, we get

$$I(s)\to I_c(s)=\int_C \frac{z^{s-1}}{e^z-1}dz \tag{57}$$

It is proved that the double complex continuation $I_c(\mathrm{s})$ is an analytic function. We have

$$z^{s-1}=e^{(a-1)\ln r-b\theta}e^{i[b\ln r+(a-1)\theta]}$$

$$=e^{(a-1)\ln r-b\theta}\ \{\cos[b\ln r+(a-1)\theta]+i\sin\}e^{i[b\ln r+(a-1)\theta]} \tag{58}$$

$$\frac{1}{e^z-1}=\frac{1}{e^{x+iy}-1}=\frac{1}{e^x\left((\cos y-1)+i\sin y\right)}$$

$$=\frac{\left((\cos y-1)-i\sin y\right)}{e^x\left((\cos y-1)+i\sin y\right)\left((\cos y-1)-i\sin y\right)}$$

$$=\frac{\cos y-1-i\sin y}{2e^x(1-\cos y)}=-\frac{1}{2e^x}+i\frac{\sin y}{2e^x(\cos y-1)} \tag{59}$$

$$\frac{z^{s-1}}{e^z-1}=\left[-\frac{1}{2e^x}+i\frac{\sin y}{2e^x(\cos y-1)}\right]\left(e^{(a-1)\ln r-b\theta}e^{i[b\ln r+(a-1)\theta]}\right)$$

$$e^{-z}z^{s-1}=e^{\square\, x+iy)}z^{(a+ib)-1}=e^{(a-1)\ln r-b\theta-x}e^{i[b\ln r+(a-1)\theta-y]} \tag{60}$$

$$=e^{(a-1)\ln r-b\theta-x}\left\{\cos(b\ln r+(a-1)\theta-y)+i\sin(b\ln r+(a-1)\theta-y)\right\}=A+iB \tag{61}$$

$$A=e^{(a-1)\ln r-b\theta-x}\cos(b\ln r+(a-1)\theta-y)$$
$$B=e^{(a-1)\ln r-b\theta-x}\sin(b\ln r+(a-1)\theta-y) \tag{62}$$

Similar to the calculations of Eqs.(47) ~ (50), it can be proved that Eq.(33) can be satisfied, so the function $I_c(s)$ described by Eq.(57) is an analytic function. Riemann used it to calculate and deduce the Zeta function equation.

V. THE COMPLEX CONTINUATION OF COMPLEMENTARY FORMULA OF GAMA FUNCTION

5.1 The complex continuation of complementary formula of the Gama function does not hold

Suppose that a is a real non-integer number with $0<a<1$, it can be proved to exist the following formula [4]

$$\int_0^\infty \frac{x^{a-1}}{x+1}dx=\frac{\pi}{\sin a\pi} \tag{69}$$

Though the formula is defined in the real field, it needs to use the Residue theorem of complex function to prove. Let $s=a+ib$, the complex continuation of Eq.(69) is considered to directly let $a\to s$ in Eq.(69) at present and get

$$\int_0^\infty \frac{x^{s-1}}{x+1}dx=\frac{\pi}{\sin s\pi} \tag{70}$$

Correspondingly, the complex continuation of Eq.(11) is considered to be

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin s\pi} \qquad 0<\mathrm{Re}(s)<1 \tag{71}$$

Eq. (71) was used when Riemann deduced the integral form of the Zeta function in his original paper in 1859.

It is proved below that Eq. (70) does not hold, so Eq.(71) does not hold too. We write the left side of Eq.(70) as

$$\int\frac{x^{s-1}}{x+1}dx=\int_0^\infty\frac{x^{a+ib-1}}{x+1}dx=\int_0^\infty\frac{x^{a-1}e^{ib\ln x}}{x+1}dx$$
$$=\int_0^\infty\frac{x^{a-1}}{x+1}\cos(b\ln x)dx+i\int_0^\infty\frac{x^{a-1}}{x+1}\sin(b\ln x)dx \tag{72}$$

According to the Euler formula, we have

$$e^{is}=e^{ia-b}=e^{-b}\left(\cos a+i\sin a\right) \tag{73}$$

$$\sin(a+ib)=\frac{e^{i(a+ib)}-e^{-i(a+ib)}}{2i}=\frac{e^{-b}(\cos a+i\sin a)-e^{b}(\cos a-i\sin a)}{2i}$$

$$=\frac{-(e^{b}-e^{-b})\cos a+i(e^{b}+e^{-b})\sin a}{2i} \tag{74}$$

$$\frac{\pi}{\sin(a+ib)\pi}=\frac{i2\pi}{-(e^{b\pi}-e^{-b\pi})\cos a\pi+i(e^{b\pi}+e^{-b\pi})\sin a\pi}$$

$$=\frac{2\pi(e^{b\pi}+e^{-b\pi})\sin a\pi}{e^{2b\pi}+e^{-2b\pi}+2(\sin^2 a\pi-\cos^2 a\pi)}-i\frac{2\pi(e^{b\pi}-e^{-b\pi})\cos a\pi}{e^{2b\pi}+e^{-2b\pi}+2(\sin^2 a\pi-\cos^2 a\pi)} \tag{75}$$

If Eq.(70) holds, by comparing the real parts and the imaginary parts of Eq.(72) and (75), we have

$$\int_0^\infty \frac{x^{a-1}}{x+1}\cos(b\ln x)dx=\frac{2\pi(e^{b\pi}+e^{-b\pi})\sin a\pi}{e^{2b\pi}+e^{-2b\pi}+2(\sin^2 a\pi-\cos^2 a\pi)} \tag{76}$$

$$\int_0^\infty \frac{x^{a-1}}{x+1}\sin(b\ln x)dx=\frac{-2\pi(e^{b\pi}+e^{-b\pi})\cos a\pi}{e^{2b\pi}+e^{-2b\pi}+2(\sin^2 a\pi-\cos^2 a\pi)} \tag{77}$$

Let $b=0$ in Eqs.(76) and (77), we get

$$\int_0^\infty \frac{x^{a-1}}{x+1}dx=\frac{4\pi\sin a\pi}{2(1+\sin^2 a\pi-\cos^2 a\pi)}=\frac{\pi}{\sin a\pi} \tag{78}$$

$$0=\frac{-4\pi\cos a\pi}{2(1+\sin^2 a\pi-\cos^2 a\pi)}=-\frac{\pi\cos a\pi}{\sin^2 a\pi} \tag{79}$$

Eq.(78) is completely the same as Eq.(69), but Eq. (79) can not hold when $a\neq(2n+1)/2$. Where is wrong? Let's analyze it below.

5.2 The correct calculation of complex continuation of complementary formula of the Gama function

At present, the Residue theorem is used to calculate Eq.(69). Let's repeat this calculation at first. Suppose that a is a real non-integer, we consider the integral of complex function below with $0\leq\arg z\leq 2\pi$ [4]

$$T(a)=\int_C z^{a-1}Q(z)dz \tag{80}$$

Here $Q(z)$ is a single value and analytic function everywhere except at several isolated singularities.

There are no singularities on the positive real axis. When $|z|\to 0$ and $|z|\to\infty$, $|z^aQ(z)|$ tends to zero consistently.

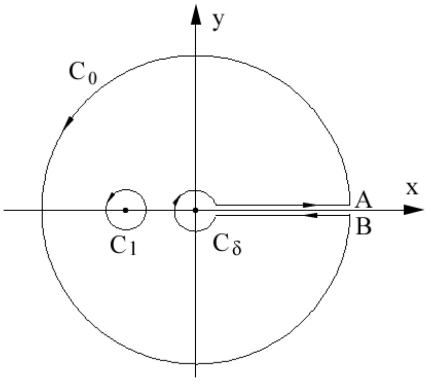

Fig.1: The contour of residue calculation

The integral contour C of residue calculation is shown in Fig.1. It stars off from the point $z = x = \delta$ on the above positive real axis, goes along the positive real axis and arrive at point A with $x = R$. Then goes along a big circle C_0 with radius and comes back to point B on the down positive real axis. Then goes along the negative direction of real axis and arrives at point $x = \delta$. At last, goes around the small circle C_δ and reaches the starting point.

Therefore, the integral of Eq.(80) can be written as

$$\int_C z^{a-1}Q(z)dz = \int_\delta^R z^{a-1}Q(z)dz + \int_{C_0} z^{a-1}Q(z)dz = \int_R^\delta z^{a-1}Q(z)dz + \int_{C_R} z^{a-1}Q(z)dz$$

$$= (1 - e^{i2a\pi})\int_\delta^R z^{a-1}Q(z)dz + \int_{C_0} z^{a-1}Q(z)dz + \int_{C_\delta} z^{a-1}Q(z)dz \qquad (81)$$

By using the Residue theorem, we have

$$\int_C z^{a-1}Q(z)dz = i2\pi \sum res\left\{z^{a-1}Q(z)\right\} \qquad (82)$$

Because of $\left|z^a Q(z)\right| \to 0$ when $|z| = \delta \to 0$ and $|z| = R \to \infty$, we have,

$$\int_{C_0} z^{a-1}Q(z)dz \to 0 \qquad \text{and} \qquad \int_{C_\delta} z^{a-1}Q(z)dz \to 0 \qquad (83)$$

Based on the formula above, we get from Eq. (82)

$$\int_\delta^R z^{a-1}Q(z)dz \to \int_0^\infty x^{a-1}Q(x)dx = \frac{i2\pi}{1 - e^{i2a\pi}} \sum res\left\{z^{a-1}Q(z)\right\} \qquad (84)$$

Let $Q(z) = 1/(1+z)$, there is an unique singularity $z = -1 = e^{i\pi}$ on the real axis. The residue is

$$\sum res\left\{z^{a-1}Q(z)\right\} = res\left\{\frac{z^{a-1}}{(1+z)'}\right\}\Bigg|_{z=e^{i\pi}} = z^{a-1}\Bigg|_{z=e^{i\pi}} = e^{i(a-1)\pi} \qquad (85)$$

Substituting it in Eq.(84), we get the current result with

$$\int_0^\infty \frac{x^{a-1}}{1+x}dx = \frac{2\pi i e^{i(a-1)\pi}}{1-e^{i2a\pi}} = \frac{i2\pi e^{-i\pi}}{e^{-ia\pi}-e^{i2a\pi}} = \frac{-i2\pi}{e^{-ia\pi}-e^{ia\pi}} = \frac{\pi}{\sin a\pi} \tag{86}$$

If let $a \to s = a+ib$, it is impossible to let $\sin a\pi \to \sin s\pi$ on the right side of Eq.(86). It needs to be calculated again. Let

$$Q_1(b,z) = \cos(b\ln z)/(1+z) \qquad Q_2(b,z) = \sin(b\ln z)/(1+z) \tag{87}$$

$$T_1(a,b) = \int_C z^{a-1}Q_1(b,z)dz \qquad T_2(a,b) = \int_C z^{a-1}Q_2(b,z)dz \tag{88}$$

When $|z| = \delta \to 0$ and $|z| = R \to \infty$, $\cos(b\ln z)$ and $\sin(b\ln z)$ are uncertain values. But we still have $|\cos(b\ln z)| \le 1$ and $|\cos(b\ln z)| \le 1$, so the conditions $|z^a Q_1(z)| \to 0$ and $|z^a Q_2(z)| \to 0$ still hold. We can still use the Residue theorem. For $T_1(a,b)$, similar to Eq.(85), we have

$$\begin{aligned}\frac{1}{i2\pi}\int_C z^{a-1}Q_1(b,z)dz &= \sum res\{z^{a-1}Q_1(z)\} = z^{a-1}\cos(b\ln z)\Big|_{z=e^{i\pi}} \\ &= e^{i(a-1)\pi}\cos(b\ln e^{i\pi}) = e^{i(a-1)\pi}\cos(ib\pi) = e^{i(a-1)\pi}\frac{e^{b\pi}+e^{-b\pi}}{2}\end{aligned} \tag{89}$$

For $T_2(a,b)$, the result is

$$\begin{aligned}\frac{1}{i2\pi}\int_C z^{a-1}Q_1(b,z)dz &= \sum res\{z^{a-1}Q_2(z)\} = z^{a-1}\sin(b\ln z)\Big|_{z=e^{i\pi}} \\ &= e^{i(a-1)\pi}\sin(b\ln e^{i\pi}) = e^{i(a-1)\pi}\sin(ib\pi) = e^{i(a-1)\pi}\frac{e^{-b\pi}-e^{b\pi}}{2i}\end{aligned} \tag{90}$$

The last result is

$$\int_0^\infty \frac{x^{a-1}}{1+x}\cos(b\ln x)dx = \frac{-i2\pi}{e^{-ia\pi}-e^{ia\pi}}\frac{e^{b\pi}+e^{-b\pi}}{2} = \frac{\pi}{\sin a\pi}\frac{e^{b\pi}+e^{-b\pi}}{2} \tag{91}$$

$$\int_0^\infty \frac{x^{a-1}}{1+x}\sin(b\ln x)dx = \frac{-i2\pi}{e^{-ia\pi}-e^{ia\pi}}\frac{e^{b\pi}+e^{-b\pi}}{2} = \frac{\pi}{\sin a\pi}\frac{e^{b\pi}-e^{-b\pi}}{2i} \tag{92}$$

When $b=0$, Eq.(91) is consistent with Eq.(78). Eq.(92) is equal to zero. The contradiction shown in Eq.(79) does not exist again. Eq.(70) becomes

$$\int_0^\infty \frac{x^{s-1}}{x+1}dx = \int_0^\infty \frac{x^{a+ib-1}}{x+1}dx = \frac{\pi}{\sin a\pi}\left[\frac{e^{b\pi}+e^{-b\pi}}{2} + i\frac{e^{b\pi}-e^{-b\pi}}{2i}\right] = \frac{\pi e^{b\pi}}{\sin a\pi} \tag{93}$$

The result of integral is a real number, rather than a complex number. The complex continuation formula (71) should be changed as

$$\Gamma(s)\Gamma(1-s)=\frac{\pi e^{b\pi}}{\sin a\pi} \qquad 0<a<1 \tag{94}$$

It is also a real number, rather than a complex number.

It should be noted that Eq. (94) holds only when $0<a<1$. Beyond this condition, Eq. (94) is still invalid. Eq. (71) was used when Riemann derived the Zeta function equation. These two results will have a great influence on the Riemann hypothesis problem.

VI. THE COMPLEX CONTINUATION OF THE GAMMA FUNCTION MULTIPLIER PRODUCT FORMULA.

The multiplier product formula of the Gama function is [4]

$$\Gamma(a)\Gamma(a+1/2)=2^{-2(2a-1)}\Gamma(2a) \qquad a>0 \tag{95}$$

According to the current theory, the complex continuation of Eq.(95) is directly written as

$$\Gamma(s)\Gamma(s+1/2)=2^{-2(2s-1)}\Gamma(2s) \tag{96}$$

Eq.(96) is regarded to be tenable on the whole complex plane. But there is no basis for this result actually. By using the definition of the Gama function, Eq.(95) can be written as

$$\int_0^\infty\int_0^\infty e^{-(x+y)}x^{a-1}y^{a-1/2}dxdy=2^{-(2a-1)}\int_0^\infty e^{-x}x^{2a-1}dx \tag{97}$$

By substituting $s=a+ib$ in Eq.(96) and considering $x^{-ib}=e^{-ib\ln x}=\cos(b\ln x)-i\sin(b\mathrm{lnx})$, the results are

$$\int_0^\infty\int_0^\infty e^{-(x+y)}x^{a-1}y^{a-1/2}\left[\cos(b\ln x)\cos(b\ln y)-\sin(b\ln x)\sin(b\ln y)\right]dxdy$$
$$+i\int_0^\infty\int_0^\infty e^{-(x+y)}x^{a-1}y^{a-1/2}\left[\sin(b\ln x)\cos(b\ln y)+\sin(b\ln y)\cos(b\ln x)\right]dxdy$$
$$=2^{-(2a-1)}\left[\cos(2b\ln 2)\int_0^\infty e^{-x}x^{2a-1}\cos(2b\ln x)dx+\sin(2b\ln 2)\int_0^\infty e^{-x}x^{2a-1}\sin(2b\ln x)dx\right]$$
$$+i2^{-(2a-1)}\left[\cos(2b\ln 2)\int_0^\infty e^{-x}x^{2a-1}\sin(2b\ln x)dx-\sin(2b\ln 2)\int_0^\infty e^{-x}x^{2a-1}\cos(2b\ln x)dx\right] \tag{98}$$

By separating the real part and the imaginary part, Eq.(98) becomes

$$\int_0^\infty\int_0^\infty e^{-(x+y)}x^{a-1}y^{a-1/2}\left[\cos(b\ln x)\cos(b\ln y)-\sin(b\ln x)\sin(b\ln y)\right]dxdy$$
$$=2^{-(2a-1)}\left[\cos(2b\ln 2)\int_0^\infty e^{-x}x^{2a-1}\cos(2b\ln x)dx+\sin(2b\ln 2)\int_0^\infty e^{-x}x^{2a-1}\sin(2b\ln x)dx\right] \tag{99}$$

$$\int_0^\infty\int_0^\infty e^{-(x+y)}x^{a-1}y^{a-1/2}\Big[\sin(b\ln x)\cos(b\ln y)+\sin(b\ln y)\cos(b\ln x)\Big]dxdy$$

$$=2^{-(2a-1)}\Big[\cos(2b\ln 2)\int_0^\infty e^{-x}x^{2a-1}\sin(2b\ln x)dx-\sin(2b\ln 2)\int_0^\infty e^{-x}x^{2a-1}\cos(2b\ln x)dx\Big] \qquad (100)$$

The status of Eqs.(99) and (97) are equivalent, but they are different when $b \neq 0$. Eq.(97) needs proof but we have no at present. And so do for Eq.(100). So the complex continuation formula of the gamma function multiplier product is generally incredible. Because the double integrals are involved in the left sides of Eq.(99) and (100), the calculations are complex, they will be discussed in a separated paper.

VII. THE COMPLEX CONTINUATION OF THE EULER FORMULA OF PRIME NUMBERS

The Euler formula of prime numbers is

$$\sum_{n=1}^{\infty} n^{-a}=\prod_{k}\left(1-p_k^{-a}\right)^{-1} \qquad (101)$$

Here $a>1$ is a real number, $n=1,\ 2,\ 3,\ 4\cdots$ is nature number and $p_k=2,\ 3,\ 5,\ 7,\cdots$ is prime number. They are all positive integers. By considering $p_k^{-a}=1/p_k^{a}<1$, we have

$$\frac{1}{1-p_k^{-a}}=1+p_k^{-a}+p_k^{-2a}+p_k^{-3a}+\cdots \qquad (102)$$

By developing the multiple products on the right side of Eq.(101), it can be written as

$$\sum_{n=1}^{\infty} n^{-a}=1+\sum_{k}p_k^{-a}+\sum_{j<k}(p_jp_k)^{-a}+\sum_{j<k<l}(p_jp_kp_l)^{-a}+\cdots$$

$$+\sum_{j<k}(p_j^{-2a}p_k^{-a})+\sum_{j<k}(p_j^{-a}p_k^{-2a})+\sum_{j<k<l}(p_j^{-2a}p_k^{-a}p_l^{-a})$$

$$+\sum_{j<k<l}(p_j^{-a}p_k^{-2a}p_l^{-a})+\sum_{j<k<l}(p_j^{-a}p_k^{-a}p_l^{-2a})+\cdots \qquad (103)$$

Riemann extended the Euler formula of primes number into the complex field and write it as

$$\sum_{n=1}^{\infty} n^{-s}=\prod_{p_k}\left(1-p_k^{-s}\right)^{-1} \qquad (104)$$

Here $s=a+ib$ is a complex number. It should note that the Riemann's complex continuation is only a hypothesis without strict proof. It is proved below that this complex continuation has no meaning in the theory of prime numbers for it does not describe the relation between prime numbers and natural numbers again.

Let $s=a+ib$ in Eq.(104), we have

$$\sum_{n=1}^{\infty} n^{-(a+ib)} = \prod_{p_k}\left(1-p_k^{-(a+ib)}\right)^{-1} \tag{105}$$

By using $n^{-ib} = e^{-ib\ln n} = \cos(b\ln n) - i\sin(b\ln n)$, the left side of Eq.(105) can be written as

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-(a+ib)} &= 1+\frac{1}{2^{a+ib}}+\frac{1}{3^{a+ib}}+\cdots+\frac{1}{n^{a+ib}}+\cdots \\ &= 1+2^{-a}\cos(b\ln 2)+3^{-a}\cos(b\ln 3)+\cdots+n^{-a}\cos(b\ln n)+\cdots \\ &\quad -i\left(2^{-a}\sin(b\ln 2)+3^{-a}\sin(b\ln 3)+\cdots+n^{-a}\sin(b\ln n)+\cdot\cdot\right)\end{aligned} \tag{106}$$

The right side of Eq.(105) becomes

$$\begin{aligned}\prod_{p_k}\left(1-p_k^{-(a+ib)}\right)^{-1} &= 1+\sum_{k} p_k^{-a}\Big[\cos(b\ln p_k)-i\sin(b\ln p_k)\Big] \\ &+\sum_{j<k}(p_j p_k)^{-a}\Big[\cos(b\ln p_j)-i\sin(b\ln p_j)\Big]\Big[\cos(b\ln p_k)-i\sin(b\ln p_k)\Big] \\ &+\sum_{j<k<l}(p_j p_k p_l)^{-a}\Big[\cos(b\ln p_j)-i\sin(b\ln p_j)\Big]\Big[\cos(b\ln p_k)-i\sin(b\ln p_k)\Big] \\ &\times\Big[\cos(b\ln p_l)-i\sin(b\ln p_l)\Big]+\cdots+\sum_{j<k}(p_j^{-2a}p_k^{-a})\Big[\cos(2b\ln p_j)-i\sin(2b\ln p_j)\Big] \\ &\times\Big[\cos(b\ln p_k)-i\sin(b\ln p_k)\Big]+\cdots\end{aligned} \tag{107}$$

By separating the real part and the imaginary part of Eq.(105), we get

$$\begin{aligned}1+2^{-a}\cos(b\ln 2)+3^{-a}\cos(b\ln 3)+\cdots+n^{-a}\cos(b\ln n)+\cdots &= 1+\sum_{k} p_k^{-a}\cos(b\ln p_k) \\ &+\sum_{j<k}(p_j p_k)^{-a}\Big[\cos(b\ln p_j)\cos(b\ln p_k)-\sin(b\ln p_j)\sin(b\ln p_k)\Big] \\ &+\sum_{j<k<l}(p_j p_k p_l)^{-a}\Big\{\Big[\cos(b\ln p_j)\cos(b\ln p_k)-\sin(b\ln p_j)\sin(b\ln p_k)\Big]\cos(b\ln p_l) \\ &+\Big[\cos(b\ln p_j)\sin(b\ln p_k)+\sin(b\ln p_j)\cos(b\ln p_k)\Big]\sin(b\ln p_l)\Big\}+\cdots \\ &+\sum_{j<k}(p_j^{-2a}p_k^{-a})\Big[\cos(2b\ln p_j)\cos(b\ln p_k)-\sin(2b\ln p_j)\sin(b\ln p_k)\Big]+\cdots\end{aligned} \tag{108}$$

$$\begin{aligned}
&2^{-a}\sin(b\ln 2)+3^{-a}\sin(b\ln 3)+\cdots+n^{-a}\sin(b\ln n)+\cdots=\sum_{k}p_k^{-a}\sin(b\ln p_k)\\
&+\sum_{j<k}(p_jp_k)^{-a}\Big[\cos(b\ln p_j)\sin(b\ln p_k)+\sin(b\ln p_j)\cos(b\ln p_k)\Big]\\
&+\sum_{j<k<l}(p_jp_kp_l)^{-a}\Big\{\Big[\cos(b\ln p_j)\sin(b\ln p_k)+\sin(b\ln p_j)\cos(b\ln p_k)\Big]\cos(b\ln p_l)\\
&+\Big[\cos(b\ln p_j)\cos(b\ln p_k)-\sin(b\ln p_j)\sin(b\ln p_k)\Big]\sin(b\ln p_l)\Big\}+\cdots\\
&+\sum_{j<k}(p_j^{-2a}p_k^{-a})\Big[\cos(2b\ln p_j)\cos(b\ln p_k)-\sin(2b\ln p_j)\sin(b\ln p_k)\Big]+\cdots
\end{aligned}\qquad(109)$$

All quantities in Eqs.(108) and (109) are real numbers. But $\sin(b\ln p_j)$ and $\cos(b\ln p_j)$ are not integers in general. They may even be irrational numbers which can not be represented by the addition, subtraction, multiplication and division of two integers. So these two formulas do not describe the relation between nature numbers and prime numbers.

The status of Eq.(108) and (103) are parallel, but they are obviously different. Eq.(101) was proved strictly by Euler. Eq.(108) was based on the complex continuation of Eq.(101) proposed by Riemann without any proof. Because Eq.(108) does not describe the relation between natural numbers and prime numbers, and contradicts with Eq.(103), it is meaningless. Eq.(109) has no basis and proof too. We has no reason to think it is correct.

Therefore, the complex continuation of the Euler formula of prime numbers has no meaning actually. The theory of prime numbers based on this formula can not be correct.

VIII. THE INFLUENCE ON THE PROBLEM OF THE RIEMANN HYPOTHESIS

8.1 The deductions of integral form of the Riemann Zeta function and the Zeta function equation

Based on the complex continuation form of the Gama function, Riemann deduced the integral form of Zeta function [1, 6]. Let $x\to nx$ in Eq.(22), the result is

$$\begin{aligned}
\Gamma(s)&=\int_0^\infty e^{-(nx)}(nx)^{s-1}d(nx)\\
&=n^s\int_0^\infty e^{-nx}x^{s-1}dx
\end{aligned}\qquad(110)$$

Taking the summation of Eq.(110), Riemann get

$$\left(\sum_{n=1}^{\infty}n^{-s}\right)\Gamma(s)=\sum_{n=1}^{\infty}\int_0^\infty e^{-nx}x^{s-1}dx=\int_0^\infty\left(\sum_{n=1}^{\infty}e^{-nx}\right)x^{s-1}dx\qquad(111)$$

Then, Riemann used the following summation formula of series (The original paper of Riemann had not provided this formula but used it actually.) [8].

$$\sum_{n=1}^{\infty} e^{-nx} = e^{-x} + e^{-2x} + e^{-3x} + \cdots = e^{-x}\left(1 + e^{-x} + e^{-2x} + e^{-3x} + \cdots\right)$$

$$= \frac{e^{-x}}{1-e^{-x}} = \frac{1}{e^{x}-1} \qquad (112)$$

By substituting Eq.(112) in Eq.(111), Riemann obtained

$$\left(\sum_{n=1}^{\infty} n^{-s}\right)\Gamma(s) = \int_0^{\infty} \frac{x^{s-1}}{e^{x}-1}dx \qquad \mathrm{Re}(s) > 1 \qquad (113)$$

To calculate Eq.(113), Riemann extended the integral of real number x to the field of complex number with $z = x + iy \in C$ and defined the function $I(z)$

$$I(s) = \int_L \frac{(-z)^{s-1}}{e^{z}-1}dz \qquad \mathrm{Re}(s) \neq 1 \qquad (114)$$

Eq. (114) is just the double complex continuation of Gama function. The definition domain of function $I(s)$ was extended into whole complex plane except at the point $\mathrm{Re}(s) = 1$, rather than original $\mathrm{Re}(s) > 1$.

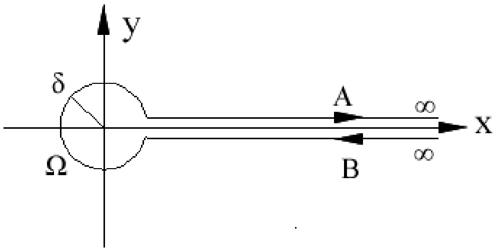

Fig. 2: The integral path of Riemann Zeta function

The path of integral started from $x \to \infty$ to $x = \delta$ along the straight line B under the x - axis. Here δ was a small quantity. Then the path was along the circle Ω with radius $\sqrt{x^2 + y^2} = \delta$ around the original point of the coordinate system. At last, the path was from $x = \delta$ to $x \to \infty$ along the straight line A above the x - axis.

According to Fig. 2, Eq.(114) contains three items.

$$I(s) = \int_{\infty}^{\delta} \frac{(-x)^{s-1}}{e^{x}-1}dx + \int_{\Omega} \frac{(-z)^{s-1}}{e^{z}-1}dz + \int_{\delta}^{\infty} \frac{(-x)^{s-1}}{e^{x}-1}dx \qquad (115)$$

Riemann's paper provided the following result without concrete calculation [1]

$$I(s) = (e^{i\pi s} - e^{-i\pi s})\int_0^{\infty} \frac{x^{s-1}}{e^{x}-1}dx \qquad (116)$$

It indicated that Riemann assumed that the medium item on the right side of Eq.(115) was zero

$$\int_{\Omega} \frac{(-z)^{s-1}}{e^{z}-1}dz = 0 \qquad (117)$$

By using the Euler's formula

$$\frac{e^{i\pi s}-e^{-i\pi s}}{2i}=\sin(\pi s) \tag{118}$$

and substituting Eqs.(116) and (118) in Eq.(113), the result was

$$I(s)=i2\sin(\pi s)\Gamma(s)\sum_{n=1}^{\infty}n^{-s} \tag{119}$$

By using the complementary formula of the Gama function

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin(\pi s)} \tag{120}$$

and considering Eq. (114), Eq. (119) can be written as

$$\sum_{n=1}^{\infty}n^{-s}=\frac{I(s)}{i2\sin(\pi s)\Gamma(s)}=\frac{\Gamma(1-s)}{2\pi i}\int_{L}\frac{(-z)^{s-1}}{e^{z}-1}dz \tag{121}$$

Riemann used the method of contour integral and the residue theorem, and obtained the integral

$$\frac{1}{2\pi i}\int_{L}\frac{(-z)^{s-1}}{e^{z}-1}dz=2(2\pi)^{s-1}\sin\frac{\pi s}{2}\sum_{n=1}^{\infty}n^{-(1-s)} \tag{122}$$

Eq.(121) becomes

$$\sum_{n=1}^{\infty}n^{-s}=2(2\pi)^{s-1}\Gamma(1-s)\sin\frac{\pi s}{2}\sum_{n=1}^{\infty}n^{-(1-s)} \tag{123}$$

By considering the original definition of the Riemann Zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s} \qquad \zeta(1-s)=\sum_{n=1}^{\infty}n^{-(1-s)} \tag{124}$$

the Riemann Zeta function equation was obtained.

$$\zeta(s)=2(2\pi)^{s-1}\Gamma(1-s)\sin\frac{\pi s}{2}\zeta(1-s) \tag{125}$$

Eq.(125) describes the relation between $\zeta(s)$ and $\zeta(1-s)$. Then, Riemann introduced the transformation [1,4]

$$\xi(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{126}$$

and proved the follow symmetry relation

$$\xi(s)=\xi(1-s) \tag{127}$$

Since the functions $\zeta(s)$ and $\xi(s)$ have the same zero, based on Eq.(126), Riemann proposed the hypothesis that all non-trivial zeros of Zeta function lie on the straight line of $s=1/2+ib$.

At present, it is generally believed that after the Riemann complex continuation, the right side of Equation (125) becomes a new definition of Zeta function, so the left side of Eq.(125) is no longer in the form of Eq. (124). However, this is not the case because to get Eq. (125) from Eq. (123), Eq. (124) must be used. It must be remembered that the prototype of Eq. (125) is Eq. (123). Equation (125) is only a symbolic simplified representation of Eq.(123). This is very important in the discussion of consistency problem of the Zeta function equation.

8.2 The problems existing in the Riemann's deduction of the Zeta function equation

There are many problems in Riemann's original paper in 1859 on the derivation of the Zeta function equation, which the author had discussed in detail in the paper [1]. The most critical one is that the middle term on the right side of Eq.(115) was missing. So Eq.(117) is not equal to zero in general. The calculation result is as follows

$$\int_{\Omega} \frac{(-z)^{s-1}}{e^{z}-1} dz \approx \frac{\delta^{a-1}}{a-1} \cos\big(\pi(a-1)\big) = \mathrm{Q}(\delta) \tag{128}$$

If $\mathrm{Re}(s)=a>1$, when $\delta \to 0$, we have $Q(\delta) \to 0$, which is the result in the Riemann's original paper. However, if $\mathrm{Re}(s)=a\le 1$, when $\delta \to 0$, we have $Q(\delta) \to \infty$. The right side of Eq.(115) becomes infinite.

Therefore, the integral form of the Riemann Zeta function does not change the divergence of its series summation form. The Riemann Zeta function equation is meaningless in the field $\mathrm{Re}(s)=a\le 1$. Since the Riemann hypothesis involved the values at the points $\mathrm{Re}(s)=1/2<1$, the result of Eq. (128) makes the Riemann hypothesis meaningless.

This result also explains why the two sides of the Riemann Zeta function equation (125) are inconsistent [2]. And why the Riemann hypothesis is so hard to be proved, because the Zeta function equation itself doesn't hold.

In the proof of Eq. (127), Riemann also used a formula of Jacobean function $\theta(x)=\sqrt{x}\theta(1/x)$. The formula is also tenable when $x>0$. If $x=0$, the formula doesn't make sense. So Eq.(127) is also problematic because the lower limit of integration is $x=0$ in the calculation process, so this formula cannot be used [1].

In addition, there is another problem that has not been found in the author's paper [1]. Here is a supplement. To derive the formula (111), the order of integral sign and summation sign needs to be exchanged. The condition of interchangeability is that the integrated function is converges uniformly [9]. Since the lower limit of integral is zero, it leads to singularity. The integrated function can not be converges uniformly at point $x=0$, so the integral sign and the summation sign in Eq. (111) cannot be exchanged. We have

$$\left(\sum_{n=1}^{\infty} n^{-s}\right)\Gamma(s) = \sum_{n=1}^{\infty}\left(\int_{0}^{\infty} e^{-nx} x^{s-1} dx\right) \neq \int_{0}^{\infty}\left(\sum_{n=1}^{\infty} e^{-nx}\right) x^{s-1} dx \tag{129}$$

Therefore, we can not obtain Eq.(113) form Eq.(111).

The key problem is caused by the use of summation formula (112), which is meaningless under conditions $x = 0$. However, the integral lower limit of the Zeta function is zero, which leads to the above problems and makes the Riemann Zeta function equation and the Riemann hypothesis meaningless.

8.3 The influence of the calculation in this paper on the Riemann Zeta function equation

The influences of the calculations in this paper on the Riemann Zeta function equation are below.

1. According to the calculation in Section 4.3, the single complex continuation of the Gama function is not an analytic function. Therefore, Eqs. (110) and (113) are practically meaningless.
2. Because Eq.(120) does not hold, we can not get Eqs.(122) and (123). Eq.(121) becomes

$$\sum_{n=1}^{\infty} n^{-s} = \frac{I(s)}{2i \sin \pi s \Gamma(s)} = \frac{\sin \pi a \Gamma(1-s)}{2i\pi e^{\pi b} \sin \pi s} \int_L \frac{(-z)^{s-1}}{e^z - 1} dz$$

$$= \frac{\sin \pi a \Gamma(1-s)}{e^{\pi b} \sin \pi s} 2(2\pi)^{s-1} \sin \frac{\pi s}{2} \sum_{n=1}^{\infty} n^{-(1-s)}$$

$$= \frac{(2\pi)^{s-1} \sin \pi a \Gamma(1-s)}{e^{\pi b} \cos(\pi s / 2)} \sum_{n=1}^{\infty} n^{-(1-s)} \tag{130}$$

Correspondingly, Eq.(125) becomes

$$\zeta(s) = \frac{(2\pi)^{s-1} \sin \pi a \Gamma(1-s)}{e^{\pi b} \cos(\pi s / 2)} \zeta(1-s) \tag{131}$$

If we still use the transformation (126), the symmetry of formula (127) does not exist, there is no symmetry of formula (127), we have $\xi(s) \neq \xi(1-s)$. Because an infinite item is missing, Eq. (131) is still inconsistent and meaningless.

If the missing of the infinite large terms is not taken into account, we suppose that Eq. (131) is correct. According to the understanding of existing theory, the right side of Eq. (131) is regarded as a redefinition of the Riemann Zeta function, then we discuss its nontrivial zero problems.

According to the standard method proposed in the author's paper [2], the real and the imaginary parts on both sides of Eq.(131) are completely separated, then comparing them. It can be strictly proved with $\zeta(1-s) \neq 0$, so the zeros of Eq.(131) are located at the points satisfying $\sin \pi a = 0$. They are $(k = 0, \pm 1, \pm 2, \cdots)$

$$a = \frac{2k+1}{2} = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2} \cdots \tag{132}$$

No matter whether they are nontrivial zeros or not, the Riemann hypothesis is proved untenable again.

IX. CONCLUSIONS

In this paper, it is proved that the negative continuation of the Gama function in the real number field is still infinite according to the direct continuation method. The reason is that the form of the Gama function has no any change in the extended field. The single complex continuation of the Gama function does not satisfy the Cauchy-Riemann equation, so it is not an analytic function. But the double complex continuation of the Gama function satisfies the Cauchy-Riemann equation, so it is an analytic function. The complex continuation of the complementary formula of the Gama function is proved to be wrong, and the correct calculation result is given in this paper.

The paper also discusses the complex continuation of the Euler prime formula, and proves that the real part of the formula is different from the original Euler prime formula after the continuation. So it no longer represents the relationship between natural numbers and prime numbers. Therefore, the complex continuation of the Euler prime number formula is meaningless in mathematics. We can not discuss the distribution of prime numbers based on it.

At the end of this paper, we discuss the influence of the calculation results on the Riemann Zeta function equation and the Riemann hypothesis, and prove again that the Riemann hypothesis does not hold from another angle.

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